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Nonintersecting String Model and Graphical Approach: Equivalence with a Potts Model

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Using a graphical method we establish the exact equivalence of the partition function of a q-state nonintersecting string (NIS) model on an arbitrary planar, even-valenced, lattice with that of a q^2-state Potts model on a related lattice. The NIS model considered in this paper is one in which the vertex weights are expressible as sums of those of basic vertex types, and the resulting Potts model generally has multispin interactions. For the square and Kagomé lattices this leads to the equivalence of a staggered NIS model with Potts models with anisotropic pair interactions, indicating that these NIS models have a first-order transition for $q > 2$. For the triangular lattice the NIS model turns out to be the five-vertex model of Wu and Lin and it relates to a Potts model with two- and three-site interactions. The most general model we discuss is an oriented NIS model which contains the six-vertex model and the NIS models of Stroganov and Schultz as special cases.

KEY WORDS: Nonintersecting string model; Potts model; vertex model; graphical approach.

1. INTRODUCTION

Great progress has been made in recent years in solving lattice models in statistical physics.[(1)] Many of the solved problems can be formulated as vertex models in which the system is described by assigning states to the lattice edges and Boltzmann weight factors to the vertices dependent on the incident states. For many of the earlier solved models the edges can be in one of two states (colors) and the configurations can be described in terms

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of strings of conserved color on the lattice edges. Included are the ice-rule models,[2,3] the eight-vertex model,[4] and the (critical) Potts model.[5,6]

Recently, there has been increasing interest in considering string models with more than two colors.[7] One of the earliest investigations is by Stroganov,[8] who considered a 3-state nonintersecting string model, a model in which the states are described by strings of conserved colors that do *not* intersect, and obtained its solution in two special cases. Stroganov's result was generalized to an arbitrary number, $q \geqslant 2$, of states by Schultz[9]; and Perk and Schultz[7,10,11] further extended the general $q \geqslant 3$ solution to $q+1$ distinct cases. (The details of the analysis together with the consideration of some additional complex cases can be found in Ref. 12.) These investigations, which have been carried out using the commuting transfer matrices approach and the matrix inversion trick, lead, quite surprisingly, to a bulk partition function identical to that of the critical Potts model (or the six-vertex model). There has been no direct simple proof of this mystifying fact which Baxter[13] referred to as "weak equivalence". In general there is only the heuristic matrix inversion argument, except for one case for which a Bethe Ansatz could be carried out.[11],[3]

In this and a forthcoming paper[14] we shall report on further exact results on the general q-state vertex problem. We shall use a graphical approach which permits us to discuss vertex models on arbitrary planar lattices. We shall establish new equivalences between lattice-statistical models and resolve, among other things, in simple graphical terms the problem concerning the weak equivalence observed above.

In this paper we start defining the general q-state vertex model on a square lattice. We shall show how it can be formulated, equivalently, as an interaction-around-a-face model. This equivalence establishes a connection between two types of lattice-statistical problems, which are often considered in different contexts. In Section 2 we shall also define the nonintersecting string (NIS) model. The particular NIS model considered in this paper is a "separable" one in which the vertex weights can be written as sums of those of basic types. In Section 3 we shall consider such a q-state NIS model on an arbitrary planar lattice of valence 4, and show that it is equivalent to a q^2-state Potts model. This equivalence can be extended to an oriented NIS model in which edges of certain colors also carry arrows. This model contains the ice-rule model as a special case when all edges are oriented. In Section 4 we shall consider a q-state NIS model on an

[3] After the completion of this research we received a preprint from T. T. Truong, who has given a proof of this weak equivalence through the consideration of the model of A. B. Zamolodchikov and M. I. Monastyrskiĭ, Zh. Eksp. Teor. Fiz. *77*: 325 (1979) [Sov. Phys. JETP *50*: 167 (1979)].

arbitrary even-valenced lattice, and show that it is also equivalent to a q^2-state Potts model, although now in general with pair as well as multispin interactions. In Section 5 we shall apply these results to regular lattices and deduce critical properties of the separable NIS model from the known properties of the Potts model.

In a later paper[14] we shall study the Baxter–Yang relation for the general NIS model. We shall verify that it is also satisfied by our general oriented NIS model for a suitable parametrization. We shall discuss implications of this, including a graphical derivation of the inversion relation and the solution of the solvable NIS models.

2. GENERAL VERTEX MODEL

2.1. Definition

In this section we consider a square lattice $\mathscr{L}$ of N sites with periodic boundary conditions. Each lattice edge of $\mathscr{L}$ can be in one of q distinct states (colors) which are specified by an edge (string) variable $\mu = 1, 2,..., q$. A vertex weight $\omega_i(\lambda, \mu, \alpha, \beta)$ is assigned to the ith vertex whose four incident edges are in respective states λ, μ, α, and β. Then, in the most general case, we have q^4 distinct vertex weights, and a q^4-vertex model. Particularly for $q = 2$, this becomes the 16-vertex model.[15] We wish to compute the per site partition function

$$\kappa = \lim_{N \to \infty} Z^{1/N} \tag{1}$$

Here Z is the partition function given by

$$Z = \sum \prod_{i=1}^{N} \omega_i(\lambda, \mu, \alpha, \beta) \tag{2}$$

where the summation is taken over all $2N$ edge configurations of the lattice and the product is taken over all N vertex weights.

2.2. Equivalence with an IRF Model

It has become customary to study lattice models utilizing the interaction-around-a-face (IRF) language for which states are assigned to lattice faces, rather than edges.[1] To establish a connection with our considerations we shall now show that the IRF model and the vertex model formulations can be seen as entirely equivalent. Specifically, we shall show that a q-state vertex model can always be transcribed into a q^2-state IRF model defined on the *same* lattice, and that, conversely, any q-state IRF model can be reformulated as a q^2-state vertex problem. Specific 1–1 mappings are given in Figs. 1a and 1b. In Fig. 1a we assign to each edge between faces with states a and b the state $\overline{ab} \equiv (a-1)\,q+b$; to all vertices with a configuration inconsistent with this assignment we give a weight $\omega=0$; if the configuration is consistent we identify the weights, i.e., $\omega(\overline{ad}, \overline{bc}, \overline{ab}, \overline{dc}) \equiv W(a, b, c, d)$. In Fig. 1b we assign to each face a state

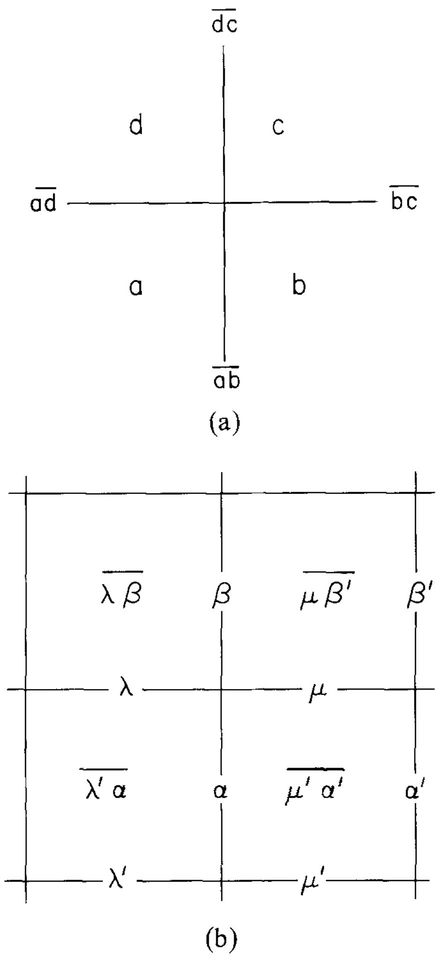

Fig. 1. (a) Configuration with Boltzmann weight $W(a, b, c, d)$ of a q-state IRF model, $a, b, c, d = 1, 2,..., q$, and the corresponding q^2-state vertex model configuration.
(b) Configuration with Boltzmann weight $\omega(\lambda, \mu, \alpha, \beta)$ of a q-state vertex model, $\lambda, \mu, \alpha, \beta = 1, 2,..., q$, and the corresponding q^2-state IRF-model configuration.

made up from the states to the right of and below the face; we assign weight $W = \omega$ or 0 depending on whether the IRF model configuration is consistent or inconsistent (following the prescription of Fig. 1b) with a vertex-model configuration.

We should note that there are, in specific models, other interesting mappings between vertex and IRF models. As examples, we mention the relation between the eight-vertex model and the Ising model with two- and four-spin interactions[15-17] and between the hard-hexagon model and a 3-state vertex model.[11]

2.3. Nonintersecting String (NIS) Model

The most general q-state model defined by (2) is a q^4-vertex model. Studies in the past,[7-12] however, have focused primarily on a subclass when the vertex weights satisfy

$$\begin{aligned} \omega_i(\lambda, \mu, \alpha, \beta) &= w^d_{\rho\rho}, && \text{if} \quad \lambda = \mu = \alpha = \beta = \rho \\ &= w^r_{\rho\sigma}, && \text{if} \quad \rho = \lambda = \alpha \neq \beta = \mu = \sigma \\ &= w^l_{\rho\sigma}, && \text{if} \quad \rho = \mu = \alpha \neq \lambda = \beta = \sigma \\ &= 0, && \text{otherwise} \end{aligned} \tag{3}$$

where the indices λ, μ, α, β are positioned as shown in Fig. 1b. In other words, only the three vertex types shown in Fig. 2 are allowed. If one now traces along lattice edges of the same color always making 90° turns, one eventually completes a loop. After this is done for all edges, the lattice is decomposed into loops which do not intersect. This is the nonintersecting string (NIS) model.[11]

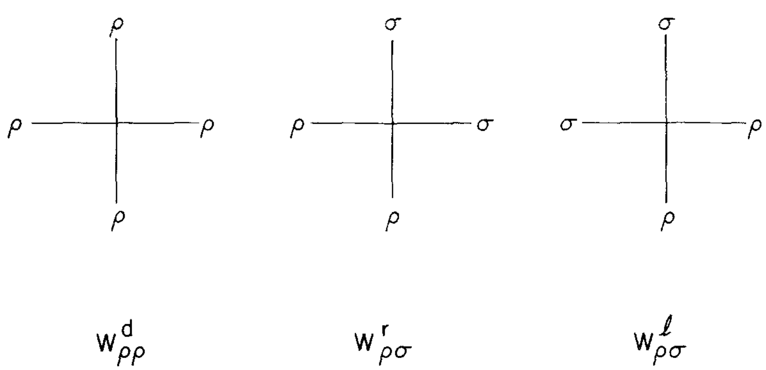

Fig. 2. The three allowed vertex types in the NIS model on a square lattice.

Generally, the restriction (3) permits $q+2q(q-1)=q(2q-1)$ distinct vertex configurations, and the q-state NIS model becomes a $q(2q-1)$-vertex model. This leads to, for $q=2$, a 6-vertex model which can be directly mapped into a staggered ice-rule model.[4] The $q=3$ problem was considered by Stroganov,[(8)] who found two soluble cases. The most general solution is by Perk and Schultz[(7,10,11)] who solved the NIS model (3) in $q+1$ distinct cases for arbitrary $q\geqslant 3$. It was found that, in all soluble cases, the solution is the same as that of the critical Potts model. In the next section we show, more generally, a special NIS model can always be formulated as a Potts model, and that this can be done for any planar lattice with arbitrary Potts interactions. Particularly, the critical Potts model on the square lattice which is exactly soluble, leads to one of the previously solved cases, and this explains the weak equivalence mentioned above.

The definition of the NIS model can be extended to any lattice which has even valences at all sites. In the general NIS model only those vertex configurations which can be decomposed into non-intersecting trajectories are allowed. Globally, the lattice is decomposed into loops of given colors, which do not intersect. Explicit examples of allowed vertex configurations will be given later for the case of valence 6.

3. EQUIVALENCE OF A NIS MODEL WITH A POTTS MODEL: ARBITRARY LATTICE OF VALENCE 4

3.1. NIS Model on a Surrounding Lattice

In this section we consider a NIS model on an arbitrary planar lattice $\mathscr{L}'$ of valence 4, which does not have to be regular. The NIS model is a $q(2q-1)$-vertex model defined by the vertex types shown in Fig. 2. Since the set of faces of an even-valenced lattice is bipartite, it is convenient to shade every other face of $\mathscr{L}'$, so that the pairs of two edges having the same label (color) will either separate or join two shaded areas at a given vertex. Then we consider a NIS model with site-dependent vertex weights w_i, $i=1, 2,\ldots, N$,

$$
\begin{aligned}
w_i &= A_i + B_i, && \text{if all 4 edges have the same color}\\
&= A_i, && \text{if the shaded areas are joined}\\
&= B_i, && \text{if the shaded areas are separated}
\end{aligned}
\tag{4}
$$

[4] The mapping can be carried out by following the prescription given by Fig. 5 of Ref. 15. The resulting ice-rule model has the weights $(w^l_{12}, w^l_{21}, w^r_{21}, w^r_{12}, w^d_{22}, w^d_{11})$ and $(w^l_{21}, w^l_{12}, w^r_{12}, w^r_{21}, w^d_{11}, w^d_{22})$ alternately, on the two sublattices I and II of the square lattice, and is soluble if $w^l_{12}=w^l_{21}$, $w^r_{12}=w^r_{21}$, $w^d_{11}=w^d_{22}$.

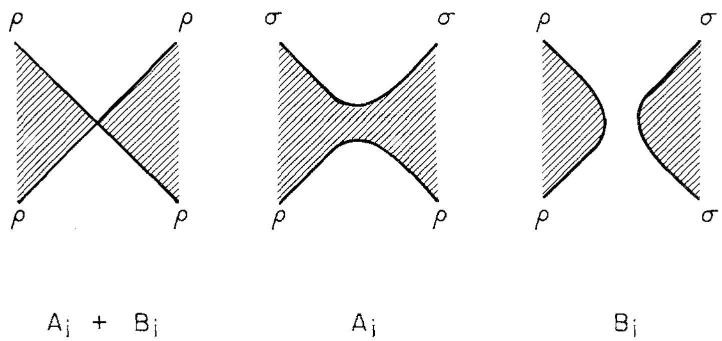

Fig. 3. Vertex weights of a NIS model on an arbitrary lattice of valence 4 ($\rho \neq \sigma$).

These situations are shown in Fig. 3. Note that in the case of a square lattice the weights w^r (and w^l) in (3) are equal to A_i and B_i alternately as i ranges from sublattice I to sublattice II, and the NIS model is therefore "staggered." Note that in (4) and Fig. 3 we can regard the vertices with weights A_i and B_i as two basic types. Then the weight of the vertex with 4 edges having the same color can be written as the sum of those of the basic types, corresponding to the two ways the vertex configuration can be decomposed. In this sense the vertex weight given by (4) is separable.

The four-coordinated lattice $\mathscr{L}'$ can be regarded as the surrounding graph (lattice) of another lattice $\mathscr{L}$ (or $\mathscr{L}_D$, the dual of $\mathscr{L}$) whose sites reside in the shaded (or unshaded) faces of $\mathscr{L}'$.[18] For planar $\mathscr{L}$ we need to pay special attention at the boundary. The boundary sites of $\mathscr{L}$ (or $\mathscr{L}_D$) are closed in by introducing "external" sites for $\mathscr{L}'$ and connecting them by straight edges. Readers are referred to Ref. 18 for examples of explicit constructions. In particular, $\mathscr{L}$ is a simple square lattice if $\mathscr{L}'$ is simple square, and $\mathscr{L}$ is either triangular (or hexagonal, the dual of triangular) if $\mathscr{L}'$ is the Kagomé lattice. At the boundary we require the edge colors be conserved at all external sites so that $\mathscr{L}'$ can again be decomposed into nonintersecting loops. While this requirement imposes a severe constraint on the vertex types that may occur at the boundary, it will not affect the bulk partition function (1) as long as the vertex weights (4) are all positive. Finally, the external sites always carry weights 1, independent of the color of the two incident edges.

3.2. Equivalence with a Potts Model

Our main result in this section is the equivalence of the q-state NIS model (4), defined on $\mathscr{L}'$, with a q^2-state Potts model defined on $\mathscr{L}$

(or $\mathscr{L}_D$). It is clear that the N sites of $\mathscr{L}'$ coincide with the N edges of $\mathscr{L}$ (or $\mathscr{L}_D$). Let the numbers of sites of $\mathscr{L}$ and $\mathscr{L}_D$ be M and M_D, respectively. The equivalence is then stated in the following theorem:

Theorem:

$$Z_{\text{NIS}}(q) = \left[\prod_{i=1}^{N} B_i\right] q^{-M} Z_{\text{Potts}}(q^2)$$
$$= \left[\prod_{i=1}^{N} A_i\right] q^{-M_D} Z_{\text{Potts}}^{(D)}(q^2) \tag{5}$$

where $Z_{\text{NIS}}(q)$ is the partition function (2) of the q-state NIS model (4), $Z_{\text{Potts}}(q^2)$ is the partition function of a q^2-state Potts model on $\mathscr{L}$ defined in the standard way[19] with edge-dependent Boltzmann weights

$$e^{K_i} = 1 + qA_i/B_i, \qquad i = 1, 2,..., N \tag{6}$$

and $Z_{\text{Potts}}^{(D)}(q^2)$ is the partition function of a q^2-state Potts model on $\mathscr{L}_D$ with Boltzmann factors

$$e^{K_i^{(D)}} = 1 + qB_i/A_i, \qquad i = 1, 2,..., N \tag{7}$$

Proof. To prove the theorem we observe that the particular separable form of the vertex weights (4) permits us to write the partition function (2) as a sum over all nonintersecting polygonal decompositions P of $\mathscr{L}'$. The summation is linear in A_i and B_i, for all i, with $A_i(B_i)$ appearing in terms where shaded areas at the ith vertex are joined (separated). The partition function takes the form

$$Z_{\text{NIS}}(q) = \sum_P q^{p(P)} \prod_i W_i(P) \tag{8}$$

where $p(P)$ is the number of polygons (loops) in P, each of which can be colored independently in q different ways. Here $W_i(P)$ is the weight of the ith site in P, equal to $A_i(B_i)$ if the shaded areas at the site are joined (separated). The expansion (8) is the key expression which leads to an equivalence to a Potts model. The theorem is proved since the partition function of a Potts model can also be written as a sum over the same polygonal decompositions.[18] The theorem now follows from a direct comparison of (8) with Eq. (9) of Ref. 18. ∎

Corollary. The NIS model (4) is also equivalent to an ice-rule model on $\mathscr{L}'$. The ice-rule model has the following vertex weights:

$$\omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6 = z^{\alpha-\gamma}B_i,\ z^{\gamma-\alpha}B_i,\ z^{\beta-\delta}A_i,\ z^{\delta-\beta}A_i,$$
$$z^{-\beta-\delta}B_i + z^{\alpha+\gamma}A_i,\ z^{\beta+\delta}B_i + z^{-\alpha-\gamma}A_i \tag{9}$$

for internal sites, and the weight

$$\omega = z^{\alpha} \tag{10}$$

for external sites. Here, α, β, γ, and δ are the relative angles spanned by the incident edges at a given site as defined in Figs. 4 and 5, the six vertex weights (9) are numbered as in Ref. 18, and

$$z^{2\pi} + z^{-2\pi} = q \tag{11}$$

The proof that the two models have identical partition functions parallels that given in Ref. 18, and will not be repeated here.

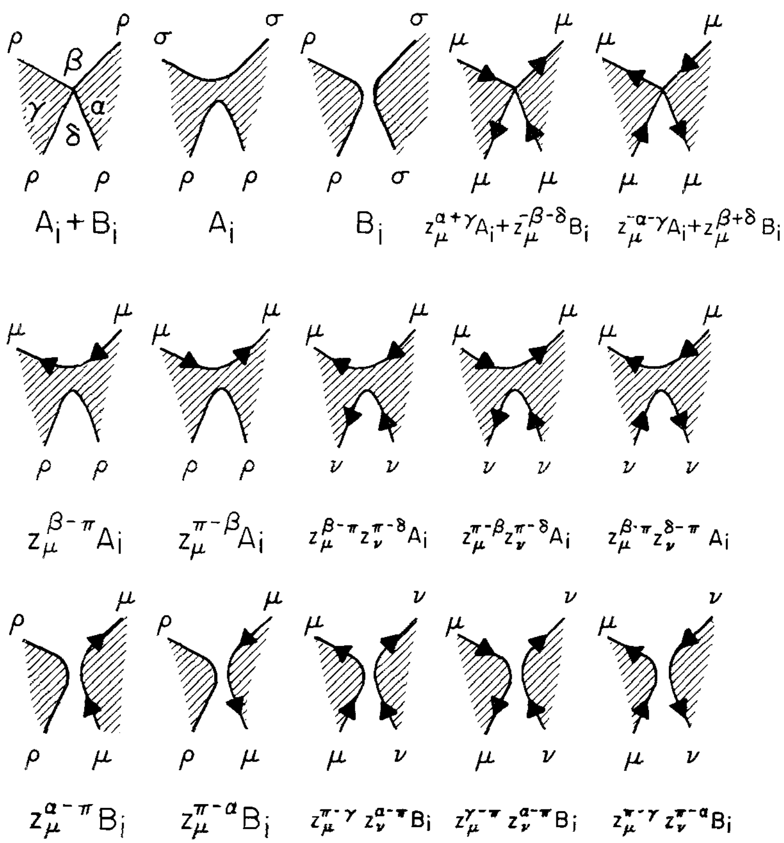

Fig. 4. The 15 topologically distinct vertex types that may occur at an interior site of an ONIS model ($\rho \neq \sigma$, $\mu \neq \nu$). The weights of the three nonbasic configurations are sums of two basic ones, noting the trivial identity $\alpha + \beta + \gamma + \delta = 2\pi$.

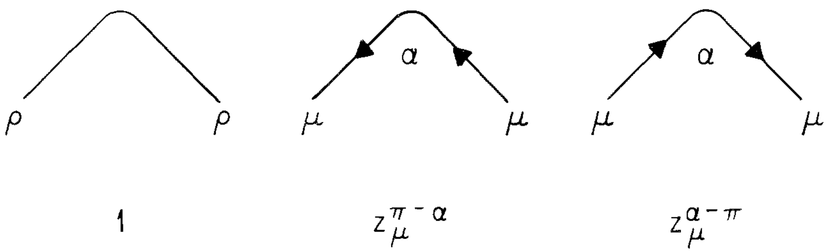

Fig. 5. The three vertex types that may occur at an external site of an ONIS model.

3.3. An Oriented NIS Model

Our proof of the theorem in the preceding section indicates that the equivalence of the NIS model (4) with a Potts model is the consequence of a local property. This observation permits an extension of the equivalence to an oriented NIS (ONIS) model.

Consider a vertex model in which the lattice edges can be colored in q_1 distinct colors and, in addition, can be colored as well as oriented (in either direction along the edge) in q_2 colors, with restriction that the numbers of in and out arrows of a given color at a vertex are the same. Again, the allowed configurations are those decomposable into altogether 15 topologically distinct vertex types that may occur at an internal site and three types that may occur at an external site. These vertex types together with their vertex weights are shown in Figs. 4 and 5. We note that the case $q_2 = 0$ is the (nonoriented) NIS model, and that the case $q_1 = 0$, $q_2 = 1$ is the ice-rule model. Following the same argument as in deriving (5), we can equal the partition function $Z_{\text{ONIS}}(q_1, q_2)$ of this ONIS model to the right-hand side of (5) or (8), provided that we take

$$q = q_1 + \sum_{\mu=1}^{q_2} (z_\mu^{2\pi} + z_\mu^{-2\pi}) \tag{12}$$

Therefore, the ONIS model defined in Figs. 4 and 5 is equivalent to a q^2-state Potts model and to a q-state (nonoriented) NIS model.

4. EQUIVALENCE OF A NIS MODEL WITH A POTTS MODEL: ARBITRARY LATTICES OF EVEN VALENCE

In this section we consider more generally a NIS model on an arbitrary planar lattice $\mathscr{L}'$ which is even valenced, i.e., the valence v_i is even for all sites i. The lattice does not have to be regular, nor does the valence

v_i need to be uniform. We shall establish that the NIS model with appropriate vertex weights is equivalent to a q^2-state Potts model.

As in Section 3, we shade every other lattice face of $\mathscr{L}'$, and the Potts model is defined on a related lattice $\mathscr{L}$ (or $\mathscr{L}_D$, the dual of $\mathscr{L}$) whose sites reside in the shaded (or unshaded) faces of $\mathscr{L}'$. The lattice $\mathscr{L}'$, which was previously introduced by one of us,[19] now serves the same role as a surrounding lattice. However, as we shall see, the related Potts model will have multispin as well as pair interactions for $v_i \geqslant 6$.

As before, we close in the boundary Potts spins by introducing new external sites with valence 2. The external sites conserve edge colors and always carry the weight 1. The appropriate assignment of vertex weights for internal sites is best illustrated by considering the case $v_i = 6$. In analogy to Fig. 3 where, for $v_i = 4$, two basic vertex types, namely, those associated with weights A_i and B_i, may occur at a vertex in a polygonal decomposition P, there are now five basic vertex types. These are the vertices with weights C_1, C_2,..., C_5 in Fig. 6. The weights of other NIS vertices with four or more edges having the same color are then written as appropriate linear combinations of all possible NIS decompositions of the vertex. There are altogether 15 distinct NIS vertex configurations, which are shown in Fig. 6 together with the associated vertex weights. We next write the partition function $Z_{\mathrm{NIS}}(q)$ in a graphical expansion in terms of polygonal decompositions P. This leads to the expression (8), but now with

$$\begin{aligned} W_i(P) &= \{A_i, B_i\}, && \text{if} \quad v_i = 4 \\ &= \{C_1, C_2, \ldots, C_5\}, && \text{if} \quad v_i = 6 \end{aligned} \tag{13}$$

If $v_i = 6$ for all i, then the NIS model is precisely the five-vertex model considered by Wu and Lin.[20] In fact, in this case Wu and Lin have shown that the NIS partition function (8), (13) is precisely the partition function of a q^2-state Potts model[5] which has two-site interactions K_1, K_2, and K_3, and a three-site interaction K for every three Potts spins surrounding a site of $\mathscr{L}'$, provided that we take

$$\begin{aligned} C_n &= e^{K_n} - 1, \qquad n = 1, 2, 3 \\ C_4 &= q \\ C_5 &= (e^{K + K_1 + K_2 + K_3} - e^{K_1} - e^{K_2} - e^{K_3} + 2)/q \end{aligned} \tag{14}$$

For details we refer to Ref. 20. If the lattice $\mathscr{L}'$ has mixed valences 4 and 6, then the NIS model is equivalent to a q^2-state Potts model using either (6)

[5] Compare (13) with Eq. (1) of Ref. 20 and use Eq. (12) of Ref. 20 after replacing q by q^2.

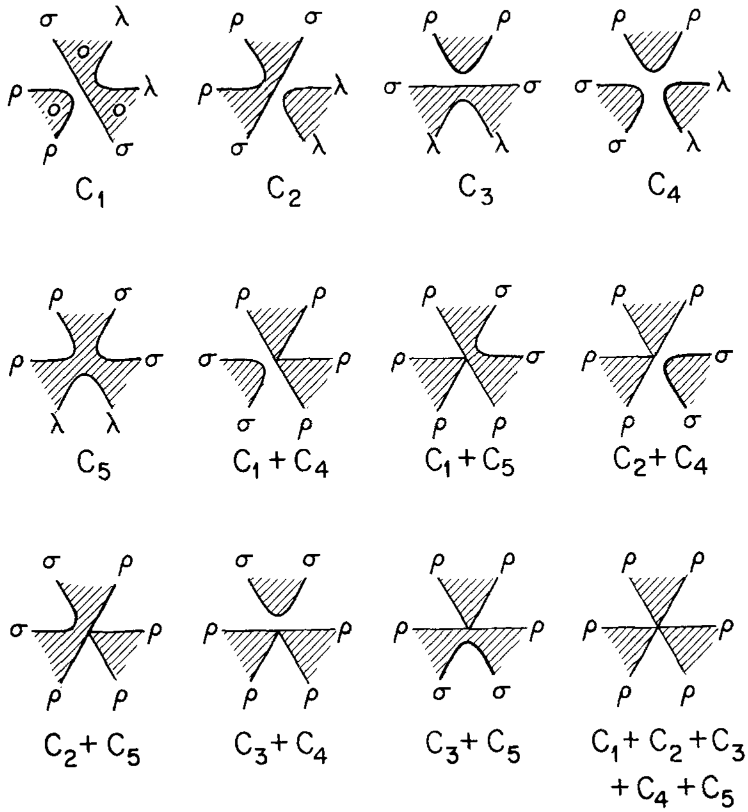

Fig. 6. NIS vertex configurations at the ith vertex of valence 6 ($\rho \neq \sigma \neq \lambda \neq \rho$, except ρ and λ may be equal in diagrams with weights C_1, C_2, and C_3). The small circles, in the first diagram, indicate the positions of the Potts spins on lattice $\mathscr{L}$.

or (14), as required by the valence, for obtaining the Potts interactions surrounding the ith site of $\mathscr{L}'$. The exact equivalence between the two partition functions is still given by (5), identifying $A_i \equiv C_5$ and $B_i \equiv C_4$ for sites of valence 6.[6]

The picture for general even valence v_i is now clear. At the ith site of $\mathscr{L}'$, there are $f(n)$ pair- and multispin Potts interactions and $g(n)$ basic NIS vertex types, where $n = v_i/2$. We have already seen that $f(2) = g(2) - 1 = 1$ and $f(3) = g(3) - 1 = 4$. Since only $g(n) - 1$ of the $g(n)$ basic vertex weights are independent, this leads to a unique determination of the Potts interactions from a given set of basic vertex weights and vice versa, for $n = 2, 3$. More generally, $g(n)$ is the number of distinct ways that

[6] This result, which holds for finite lattices, is more general than that given in Ref. 20, which assumes the thermodynamic limit.

a linear array of $2n$ points can be connected pairwisely by n nonintersecting lines which remain on one side of the array. This number has been computed by Temperley and Lieb,[21] and is given by

$$g(n) = \frac{1}{n+1} \binom{2n}{n} \tag{15}$$

The number of distinct Potts interactions is

$$f(n) = \sum_{r=2}^{n} \binom{n}{r} = 2^n - n - 1 \tag{16}$$

As in the cases of $n = 2$ and 3 we choose all vertex weights to be appropriate linear combinations of the $g(n)$ weights of the basic vertex types, according to the possible decompositions of the vertex configuration (into basic types). Then, the partition function of the NIS model is written in the form of the polygonal expansion (8), with $W_i(P)$ ranging over these $g(n)$ basic weights. To convert the polygonal decomposition expansion into a Potts partition function, we note that, quite generally,

$$g(n) - 1 > f(n), \qquad n \geqslant 4 \tag{17}$$

so that we can always equate the $g(n)$ basic vertex weights with Potts Boltzmann factors involving $f(n)$ Potts interactions. This leads to a unique determination of the Potts interactions, provided that the $g(n)$ weights are constrained for $n \geqslant 4$. So we have an equivalence of the q-state NIS model on an even-valenced lattice with a q^2-state Potts model. This equivalence can again be generalized to an ONIS model with appropriate vertex weights.

5. EXACT SOLUTIONS FOR THE SEPARABLE NIS MODEL

In the preceding sections we have established the equivalence of a q-state NIS model (on an arbitrary planar even-valenced lattice) with a q^2-state Potts model. This equivalence makes it possible to deduce properties of the NIS model from known solutions of the Potts model for regular lattices.

While the NIS model is defined for integer values of q, the particular model considered in this paper, for which the vertex weights are separable, permits a natural continuation of the partition function, through the polygonal expansion (8), to noninteger values of q. We can now discuss its critical properties.

5.1. Square Lattice

Consider a q-state staggered NIS model on a square lattice whose weights are given by (4) and Fig. 3 with

$$\begin{aligned} \{A_i, B_i\} &= \{A_1, B_1\}, \qquad i \in \text{sublattice I} \\ &= \{A_2, B_2\}, \qquad i \in \text{sublattice II} \end{aligned} \tag{18}$$

or, in terms of the vertex weights (3) and Fig. 2,

$$\begin{aligned} \{w^r_{\rho\sigma}, w^l_{\rho\sigma}\} &= \{A_1, B_1\}, \qquad i \in \text{sublattice I} \\ &= \{B_2, A_2\}, \qquad i \in \text{sublattice II} \\ w^d_{\rho\rho} &= A_i + B_i \end{aligned} \tag{19}$$

Theorem (5) now relates the partition function of the NIS model with that of an anisotropic q^2-state Potts model with interactions (6) or (7).

From known properties of the Potts model[19,22] we have the following:

(i) The NIS model is exactly soluble for $q = \sqrt{2}$ for which it becomes an Ising model.

(ii) The NIS model exhibits a continuous transition for $1 < q \leqslant 2$, with the critical exponents varying continuously with the value of q.

(iii) The NIS model exhibits a first-order transition for $q > 2$ accompanied by a nonzero latent heat which can be computed.

The anisotropic Potts model is exactly soluble[1,5] at its critical point

$$A_1 A_2 = B_1 B_2 \tag{20}$$

If we divide all vertex weights on sublattice I and II by A_1 and B_2, respectively, then the NIS model has uniform weights and becomes the soluble case NIS1 considered by Perk and Schultz.[11] Thus, we have a simple proof that the NIS1 model free energy of Ref. 11 is identical to that of the critical Potts model. This is the weak equivalence referred to in Ref. 13. Explicit expressions of the critical free energy will be discussed in a later paper.[14]

5.2. Kagomé Lattice

Consider a q-state NIS model on the Kagomé lattice with vertex weights given by (4) and Fig. 3, with

$$\begin{aligned}\{A_i, B_i\} &= \{A_1, B_1\}, \qquad i \in \text{sublattice I}\\ &= \{A_2, B_2\}, \qquad i \in \text{sublattice II}\\ &= \{A_3, B_3\}, \qquad i \in \text{sublattice III}\end{aligned} \tag{21}$$

Then Theorem (5) relates its partition function to that of an anisotropic q^2-state Potts model on a triangular (honeycomb) lattice with interactions (6) and (7). The same general conclusions on its critical properties can now be drawn as in the case of the square lattice. The NIS model is exactly soluble at its critical point

$$q + \frac{B_1}{A_1} + \frac{B_2}{A_2} + \frac{B_3}{A_3} = \frac{B_1 B_2 B_3}{A_1 A_2 A_3} \tag{22}$$

obtained from the corresponding Potts critical point.[6,23] The solution has been given in Ref. 6.

5.3. Triangular Lattice

Consider a q-state NIS model on a triangular lattice with vertex weights given by Fig. 6. Our analysis in Section 4 relates its partition function to that of a triangular Potts model with two- and three-spin interactions for every three spins surrounding an up-pointing triangle. The Potts interactions K_1, K_2, K_3, and K are uniquely determined from (14). The NIS model is self-dual and is critical at its self-dual point[6,24]

$$C_4 = C_5 \tag{23}$$

6. SUMMARY

We have established the exact equivalence of the partition function of a separable q-state NIS model on any even-valenced planar lattice with that of a q^2-state Potts model on a related lattice. The equivalence also holds for the partition function of a generalized separable q-state NIS model for which the lattice edges can be oriented. The resulting Potts model generally has multispin interactions. Critical properties of the NIS model are derived from this equivalence and the known properties of the Potts model. This includes a previously solved NIS model on a regular square lattice, now identified as the exact solution of the Potts model at the critical point.

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